

Symplectic field theory of a disk, quantum integrable systems, and Schur polynomials

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Abstract. We consider commuting operators obtained by quantization of Hamiltonians of the Hopf (aka dispersionless KdV) hierarchy. Such operators naturally arise in the setting of Symplectic Field Theory (SFT). A complete set of common eigenvectors of these operators is given by Schur polynomials. We use this result for computing the SFT potential of a disk.

Keywords. Quantum integrable systems, Symplectic field theory, Schur polynomials.

1. Introduction

1.1. Hopf integrable hierarchy and its quantization

Hopf equation

$$u_t + u u_x = 0 \quad (1.1)$$

can be considered as the spatially one-dimensional analogue of the Euler equations of motion of ideal incompressible fluid. It is perhaps the simplest example of an integrable nonlinear PDE. In the community of experts in integrable systems it is often called dispersionless Korteweg–de Vries (KdV) equation; sometimes it is also called (inviscid) Burgers equation. Eq. (1.1) is an infinite-dimensional analogue of a Hamiltonian dynamical system

$$u_t + \partial_x \frac{\delta H}{\delta u(x)} = 0, \quad H = \frac{1}{2\pi} \int_0^{2\pi} \frac{u^3(x)}{6} dx$$

(to be more specific let us consider (1.1) as a dynamical system on the space of smooth 2π -periodic functions). It possesses a complete family of pairwise commuting conservation laws

$$H_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{u^{n+2}(x)}{(n+2)!} dx, \quad n \geq -1 \quad (1.2)$$

$$\begin{aligned} \{H_n, H_m\} &:= \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta H_n}{\delta u(x)} \partial_x \left(\frac{\delta H_m}{\delta u(x)} \right) dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} \partial_x \left(\frac{u^{m+n+2}(x)}{(n+1)! m! (m+n+2)} \right) dx = 0. \end{aligned}$$

The Hopf Hamiltonian coincides with the functional H_1 ; the Hamiltonian H_0 generates spatial translations; the very first Hamiltonian H_{-1} is a Casimir of the degenerate Poisson bracket defined by the operator ∂_x . A convenient generating function of the commuting Hamiltonians can be chosen in the form

$$\mathcal{H}(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{z u(x)} dx = 1 + \sum_{n \geq -1} H_n z^{n+2}. \quad (1.3)$$

In general the procedure of quantization of a Hamiltonian system on a symplectic manifold converts functions on the manifold into operators acting on a suitable space of states. It depends on the choice of canonical coordinates on the phase space.

In our case one can use the Fourier coefficients of the 2π -periodic function

$$u(x) = \sum_{n \in \mathbb{Z}} u_n e^{i n x}$$

as coordinates on the phase space. Their Poisson brackets read

$$\{u_n, u_m\} = i n \delta_{m, -n}.$$

Thus one can choose positive Fourier coefficients as canonical coordinates on the symplectic leaves $u_0 = \text{const}$

$$q_n = u_n, \quad n \geq 1.$$

Negative Fourier coefficients will be denoted

$$p_n = u_{-n}, \quad n \geq 1.$$

Their Poisson brackets have the canonical form up to a constant factor

$$\{p_n, q_m\} = -i n \delta_{mn}, \quad \{u_0, q_n\} = \{u_0, p_n\} = 0. \quad (1.4)$$

After the quantization one has to replace the variables q_n, p_m by operators \hat{q}_n, \hat{p}_m satisfying the commutation relations

$$[\hat{p}_n, \hat{q}_m] = \hbar n \delta_{mn}. \quad (1.5)$$

Here \hbar is an independent parameter of the quantization procedure called Planck constant.

The operators \hat{q}_k, \hat{p}_k admit the following realization on the space of polynomials in the variables q_1, q_2, \dots

$$\hat{q}_k f(q) = q_k f(q), \quad \hat{p}_k f(q) = \hbar k \frac{\partial}{\partial q_k} f(q), \quad k = 1, 2, \dots \quad (1.6)$$

We can associate certain operators acting on the same space of polynomials also with the Hamiltonians H_n applying the procedure of normal

ordering. That means that, in any monomial in the p - and q -variables all p_n -s must be written on the right and, then, replaced by operators $\hat{p}_n = \hbar n \frac{\partial}{\partial q_n}$. Denote

$$\hat{H}_n^0 = \frac{1}{2\pi} : \int_0^{2\pi} \frac{u^{n+2}(x)}{(n+2)!} dx : \quad (1.7)$$

the resulting *normally ordered* quantum operators. For example,

$$\hat{H}_{-1}^0 = u_0$$

$$\hat{H}_0^0 = \frac{1}{2\pi} : \int_0^{2\pi} \frac{u^2}{2} dx := \frac{u_0^2}{2} + \sum_{n \geq 1} \hat{q}_n \hat{p}_n$$

$$\hat{H}_1^0 = \frac{1}{2\pi} : \int_0^{2\pi} \frac{u^3}{6} dx := \frac{u_0^3}{6} + u_0 \sum \hat{q}_k \hat{p}_k + \frac{1}{2} \sum_{i,j} (\hat{q}_i \hat{q}_j \hat{p}_{i+j} + \hat{q}_{i+j} \hat{p}_i \hat{p}_j)$$

etc. In the semiclassical limit $\hbar \rightarrow 0$ the operators are replaced by their symbols clearly coinciding with the original Hamiltonians H_n .

However, these quantum operators do not commute any more; this can be seen already in the commutator

$$[\hat{H}_1^0, \hat{H}_2^0] = \frac{\hbar^2}{8} \sum i j (i+j) (\hat{q}_{i+j} \hat{p}_i \hat{p}_j - \hat{q}_i \hat{q}_j \hat{p}_{i+j}) \neq 0.$$

We arrive at the following quantization problem.

Definition 1.1. Quantization of the integrable system (1.1) is a family of pairwise commuting operators \hat{H}_n , $n \geq -1$, acting on the Fock space of polynomials $\mathbb{C}[q_1, q_2, \dots]$ and, in the semiclassical limit $\hbar \rightarrow 0$, coinciding with H_n given by (1.2).

Such quantization problem¹ was addressed in [19] where a recursion procedure for constructing commuting quantum Hamiltonians has been developed. Independently a closed formula for the commuting quantum Hamiltonians was found by Y. Eliashberg [7], [8], see also [20], in the setting of Symplectic Field Theory (see below). This formula plays the central role in the present paper.

Proposition 1.2. *The operators \hat{H}_n , $n \geq -1$ defined by their generating function*

$$\hat{\mathcal{H}}(z, u_0, \hbar) = \frac{1}{2\pi s (\hbar^{1/2} z)} : \int_0^{2\pi} e^{z s (i \hbar^{1/2} z \partial_x) u(x)} dx := 1 + \sum_{n \geq -1} \hat{H}_n z^{n+2} \quad (1.8)$$

where

$$u(x) = u_0 + \sum_{k \geq 1} (\hat{q}_k e^{ikx} + \hat{p}_k e^{-ikx})$$

¹Quantization of the KdV equation equipped with the *second* Poisson bracket was first considered in a series of papers by V. Bazhanov, S. Lukyanov and A. Zamolodchikov [4]. More recently the problem of quantization of the intermediate long wave equation was addressed in [5].

and

$$s(t) = \frac{\sinh \frac{t}{2}}{\frac{t}{2}}$$

commute pairwise

$$[\hat{H}_n, \hat{H}_m] = 0, \quad n, m \geq -1.$$

Remark 1.3. The commuting operators \hat{H}_n were recently rediscovered in essentially equivalent form in [1], [11].

1.2. Schur polynomials and eigenvectors of the quantum Hamiltonians

Using the Taylor expansion of the function $s(t)$

$$s(t) = 1 + \sum_{n \geq 1} \frac{t^{2n}}{2^{2n}(2n+1)!}$$

one can spell out the formula (1.8) as follows

$$\hat{\mathcal{H}}(z, u_0, \hbar) = \frac{1}{2\pi s(\hbar^{1/2}z)} : \int_0^{2\pi} e^{z u - \frac{\hbar z^3}{24} u'' + \frac{\hbar^2 z^5}{1920} u^{(4)} - \dots} dx :$$

so, in the semiclassical limit $\hbar \rightarrow 0$, one returns to the Hamiltonians (1.2) but, starting from \hat{H}_2 there are nontrivial corrections

$$\begin{aligned} \hat{H}_{-1} &= u_0 \\ \hat{H}_0 &= \frac{1}{2\pi} : \int_0^{2\pi} \frac{u^2}{2} dx : - \frac{\hbar}{24} \\ \hat{H}_1 &= \frac{1}{2\pi} : \int_0^{2\pi} \frac{u^3}{6} dx : - \frac{\hbar}{24} u_0 \\ \hat{H}_2 &= \frac{1}{2\pi} : \int_0^{2\pi} \left[\frac{u^4}{24} + \frac{\hbar}{24} \left(u u'' - \frac{1}{2} u^2 \right) \right] dx : + \frac{7\hbar^2}{5760}. \end{aligned}$$

Observe that, in the representation (1.6) the nontrivial part of \hat{H}_0 coincides with the degree operator, up to an additive constant,

$$\hat{H}_0|_{u_0=0} = \hbar \sum_n n q_n \frac{\partial}{\partial q_n} - \frac{\hbar}{24}$$

and \hat{H}_1 with the cut-and-join operator

$$\hat{H}_1|_{u_0=0} = \frac{1}{2} \sum_{i,j} \left(\hbar (i+j) q_i q_j \frac{\partial}{\partial q_{i+j}} + \hbar^2 i j q_{i+j} \frac{\partial^2}{\partial q_i \partial q_j} \right).$$

It was observed in [13] that Schur polynomials are eigenvectors² of the cut-and-join operator. It turns out that the same is true for all commuting

²It can also be deduced from the description [3] in terms of Jack polynomials of eigenstates of the Calogero–Sutherland operator at the limit $\beta \rightarrow 1$. I thank P. Wiegman for this reference.

Hamiltonians (1.8). In order to formulate the precise statement³ let us remind basic definitions of the theory of Schur polynomials.

Introduce polynomials $h_k(q)$ by

$$\sum_{k \geq 0} h_k(q) z^k = e^{\sum_{k \geq 1} q_k \frac{z^k}{k}}, \quad h_k = 0 \quad \text{for } k < 0. \quad (1.9)$$

For any Young tableau $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0)$ (the number n for the tableau λ will be denoted $l(\lambda)$) define a polynomial

$$s_\lambda(q) = \det (h_{\lambda_i - i + j}(q))_{1 \leq i, j \leq l(\lambda)}.$$

They coincide with the standard Schur polynomials $s_\lambda(x)$, up to a normalization of the independent variables $x_k = \frac{q_k}{k}$.

Theorem 1.4. *For any partition λ the polynomial $s_\lambda\left(\frac{q}{\hbar^{1/2}}\right)$ is an eigenvector of the operator (1.8)*

$$\hat{\mathcal{H}}(z, u_0, \hbar) s_\lambda\left(\frac{q}{\hbar^{1/2}}\right) = E(z, \lambda, u_0, \hbar) s_\lambda\left(\frac{q}{\hbar^{1/2}}\right) \quad (1.10)$$

$$E(z, \lambda, u_0, \hbar) = \hbar^{1/2} z e^{z u_0} \sum_{i=1}^{\infty} e^{z \hbar^{1/2} (\lambda_i - i + \frac{1}{2})}, \quad \text{Re } z > 0.$$

The sum in the right-hand side makes sense for any infinite sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \dots$ such that $\lambda_i = 0$ for $i \gg 0$. Observe that the eigenvalue (1.10) essentially coincides with the Chern character of the partition λ introduced in [15]

$$E(z, \lambda, u_0, \hbar = 1) = z \text{ch}_\lambda(u_0, z).$$

All Schur polynomials $s_\lambda(q)$ of degree $n = |\lambda|$ form a basis in the space $V_n \subset \mathbb{C}[q]$ of all graded homogeneous polynomials of degree n in the variables q_1, \dots, q_n , where the degree is assigned according to the following rule

$$\deg q_i = i, \quad i = 1, 2, \dots$$

Thus, the theorem 1.4 provides us with a complete family of eigenstates of the commuting quantum Hamiltonians.

1.3. Applications to Symplectic Field Theory

Symplectic Field Theory (SFT) is an approach to constructing topological invariants of contact manifolds and symplectic cobordisms between them by enumeration of pseudoholomorphic maps of open Riemann surfaces with certain boundary conditions. It also provides one with a surgery tool in computing Gromov–Witten (GW) invariants of closed symplectic manifolds by gluing them from simple blocks.

With a compact contact⁴ manifold V the SFT associates a bosonic Fock space $\mathcal{F}(V)$ generated by closed Reeb orbits on V , assuming them to be all

³A closely related result was recently announced in [1].

⁴This assumption can be slightly relaxed, see [8] for details.

nodegenerate. Choosing a closed odd-dimensional differential form θ on V one obtains an infinite sequence of pairwise commuting quantum Hamiltonians \hat{H}_k^V acting on the Fock space. For the particular case $V = S^1$ this gives [7, 8] the quantum Hamiltonians (1.8), see more details in Section 3 below.

A symplectic cobordism W between two contact manifolds V_+ and V_- ,

$$\partial W = V_+ \cup (-V_-)$$

defines an operator

$$Z_{\text{SFT}}^W : \mathcal{F}(V_+) \rightarrow \mathcal{F}(V_-) \quad (1.11)$$

called the total SFT potential of W . For a closed symplectic manifold W , $\partial W = \emptyset$, the SFT potential coincides with the total Gromov–Witten (GW) descendent potential of W .

In particular, to a symplectic manifold W with the boundary V the total SFT potential of W defines a state Ψ_{SFT}^W in the Fock space $\mathcal{F}(V)$ (or in its dual, depending on the orientation of $V = \pm \partial W$). A suitable choice of an even-dimensional closed differential form $\phi = \theta \wedge d\rho$ on W makes the SFT potential Ψ_{SFT}^W depend on an infinite number of coupling parameters t_k^ϕ . The dependence on these parameters is determined from a system of pairwise commuting Schrödinger⁵ equations

$$\hbar \frac{\partial}{\partial t_k^\phi} \Psi_{\text{SFT}}^W = \hat{H}_k^V \Psi_{\text{SFT}}^W, \quad k = 0, 1, \dots$$

Our goal is to compute this SFT potential for the simplest example $V = S^1$, $W = \text{disk}$.

Denote

$$E(z, \lambda, u_0, \hbar) = 1 + \sum_{n \geq -1} E_n(\lambda, u_0, \hbar) z^{n+2} \quad (1.12)$$

the Taylor expansion of the eigenvalues (1.10), see the explicit formula (2.13) below.

Theorem 1.5. *The total SFT potential of a disk $\subset \mathbb{C}$ is given by the following formula*

$$\Psi_{\text{SFT}}^{\text{disk}}(u_0, \mathbf{t}, \mathbf{p}; \hbar) = \sum_{\lambda} \hbar^{-\frac{|\lambda|}{2}} \frac{\dim \lambda}{|\lambda|!} e^{\frac{1}{\hbar} \sum_{k \geq 0} t_k E_k(u_0, \hbar, \lambda)} s_{\lambda} \left(\frac{p}{\hbar^{1/2}} \right). \quad (1.13)$$

Collecting in (1.13) partitions of a given number $n = |\lambda|$ one obtains a part of the SFT potential corresponding to maps of Riemann surfaces with cylindrical ends of a given degree n . For low degrees $|\lambda| \leq 3$ keeping the variables t_k for $k \leq 3$ one obtains (for simplicity we set $u_0 = 0$; recall that

⁵Due to our normalization of canonical coordinates and Hamiltonians the $\sqrt{-1}$ does not appear in the equations.

this parameter is responsible for adding punctures on the Riemann surface with no cohomology classes inserted)

$$\begin{aligned} \Psi_{\text{SFT}}^{\text{disk}} = & e^{-\frac{t_0}{24} + \frac{7\hbar t_2}{5760}} \left[1 + \frac{1}{\hbar} e^{t_0 + \frac{\hbar t_2}{24}} p_1 \right. \\ & + \frac{1}{2\hbar^2} \left(e^{2t_0 + \hbar^{\frac{1}{2}} t_1 + \frac{7\hbar t_2}{12} + \frac{5}{24} \hbar^{\frac{3}{2}} t_3} (p_1^2 + \hbar^{\frac{1}{2}} p_2) + e^{2t_0 - \hbar^{\frac{1}{2}} t_1 + \frac{7\hbar t_2}{12} - \frac{5}{24} \hbar^{\frac{3}{2}} t_3} (p_1^2 - \hbar^{\frac{1}{2}} p_2) \right) \\ & + \frac{1}{36\hbar^3} \left(e^{3t_0 + 3\hbar^{\frac{1}{2}} t_1 + \frac{21}{8} \hbar t_2 + \frac{13}{8} \hbar^{\frac{3}{2}} t_3} (p_1^3 + 3\hbar^{\frac{1}{2}} p_1 p_2 + 2\hbar p_3) + 4e^{3t_0 + \frac{9}{8} \hbar t_2} (p_1^3 - \hbar p_3) \right. \\ & \left. \left. + e^{3t_0 - 3\hbar^{\frac{1}{2}} t_1 + \frac{21}{8} \hbar t_2 - \frac{13}{8} \hbar^{\frac{3}{2}} t_3} (p_1^3 - 3\hbar^{\frac{1}{2}} p_1 p_2 + 2\hbar p_3) \right) \right] + \dots \end{aligned}$$

It is not difficult to see that the resulting expansion contains only integer powers of Planck constant due to the following well known properties (see, e.g., [14])

$$\begin{aligned} s_{\lambda'}(p) &= (-1)^{|\lambda|} s_{\lambda}(-p) \\ \dim \lambda' &= \dim \lambda. \end{aligned}$$

Here λ' is the transposed Young tableau.

Remark 1.6. Setting in (1.13) $u_0 = 0$, $t_1 = \beta$, $t_k = 0$ for $k \neq 1$ one obtains the I.P. Goulden and D.M. Jackson generating function for Hurwitz numbers [10] (see also [21]) in the representation of M. Kazarian and S. Lando [13].

Corollary 1.7. *The tau-function $\tau(\mathbf{p}, \epsilon) = \Psi_{\text{SFT}}^{\text{disk}}(\mathbf{t}, \mathbf{p}, u_0, \epsilon^2)$ for arbitrary values of the parameters u_0 and $\mathbf{t} = (t_0, t_1, \dots)$ satisfies equations of the KP hierarchy*

$$\sum_{j \geq 0} h_j(-2\mathbf{y}) h_{j+1}(\epsilon \tilde{D}) e^{\epsilon \sum_{k \geq 1} y_k D_k} \tau \cdot \tau = 0 \quad \text{for all } \mathbf{y} = (y_1, y_2, \dots). \quad (1.14)$$

Here

$$D = (D_1, D_2, \dots)$$

are the Hirota bilinear operators,

$$D_k \tau \cdot \tau = \left[\frac{\partial}{\partial q} \tau(p_k + q) (\tau(p_k - q)) \right]_{q=0},$$

$$\tilde{D} = (D_1, 2D_2, 3D_3, \dots).$$

Recall that, expanding eqs. (1.14) with respect to the indeterminates y_1 , y_2 , etc. one obtains an infinite family of bilinear equations for tau-function of the KP hierarchy. E.g., the first two equations read

$$\begin{aligned} (12D_2^2 - 12D_1D_3 + \hbar D_1^4) \tau \cdot \tau &= 0 \\ (6D_2D_3 - 6D_1D_4 + \hbar D_1^3D_2) \tau \cdot \tau &= 0 \end{aligned}$$

(recall: here and below $\epsilon^2 = \hbar$). The second logarithmic derivative

$$u = \epsilon^2 \partial_x^2 \log \tau$$

as function of $x = p_1$, $y = p_2$, $t = p_3$ satisfies the Kadomtsev–Petviashvili equation

$$u_{xt} = u_{yy} + \left(u u_x + \frac{\hbar}{12} u_{xxx} \right)_x. \quad (1.15)$$

Following the general scheme of [9] one can use the SFT potential of the disk for computation of the stationary sector (in the terminology of [18]) of Gromov–Witten theory of projective line \mathbf{P}^1 . Namely,

Corollary 1.8. *Denote $\overline{M}_{g,n}^\bullet(\mathbf{P}^1, d)$ the space⁶ of degree d stable maps to \mathbf{P}^1 of not necessarily connected algebraic curves of the total genus g with n punctures. Let*

$$\chi : \overline{M}_{g,n}^\bullet(\mathbf{P}^1, d) \rightarrow \mathbb{Z}$$

be a locally constant function taking value $\chi = 2g - 2k$ on the subset of stable maps of k -component curves. Then

$$\begin{aligned} Z_{\mathbf{P}^1}^\bullet(\mathbf{t}, z, \hbar) &:= \sum_{g \geq 0} \sum_{d \geq 0} z^d \sum_{m, n} \frac{u_0^m}{m!} \sum_{k_1, \dots, k_n} \frac{t_{k_1} \dots t_{k_n}}{n!} \int_{[\overline{M}_{g, n+m}^\bullet(\mathbf{P}^1, d)]^{\text{virt}}} \hbar^{\frac{\chi}{2}} \text{ev}_1^*(\omega) \dots \text{ev}_n^*(\omega) \psi_1^{k_1} \dots \psi_n^{k_n} \\ &= \sum_{\lambda} \left(\frac{z}{\hbar} \right)^{|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 e^{\frac{1}{\hbar} \sum_{k \geq 0} t_k E_k(\lambda, u_0, \hbar)}. \end{aligned} \quad (1.16)$$

In this formula $\omega \in H^2(\mathbf{P}^1)$ is the area form normalized by $\int_{\mathbf{P}^1} \omega = 1$.

Specializing (1.16) at $u_0 = 0$, $\hbar = 1$ and collecting the terms of degree d one arrives at eq. (0.25) of the A. Okounkov and R. Pandharipande paper [18] playing an important role in their analysis of GW/Hurwitz correspondence.

The paper is organised as follows. In Section 2 we recall basics about the so-called boson-fermion correspondence. We apply this technique for proving Theorem 1.4. In Section 3 we very briefly remind main definitions of Symplectic Field Theory and prove Theorem 1.5 as well as Corollaries 1.7 and 1.8.

2. Boson-fermion correspondence and proof of Theorem 1.4

For simplicity let us do the calculations setting $\hbar = 1$. The original normalization will be restored at the end of the proof.

We will use the so-called boson-fermion correspondence. Let us briefly remind the main step of this construction; the reader can find more details in [16], [2]. By definition the algebra of free bosons has generators⁷ q_n , p_n , $n \geq 1$, satisfying commutation relations

$$[p_n, q_m] = n \delta_{n,m}, \quad [p_n, p_m] = [q_n, q_m] = 0. \quad (2.1)$$

⁶Here we follow notations of [18].

⁷Hats over the bosonic operators will be omitted in this section.

It admits a natural representation on the bosonic Fock space

$$\mathbb{C}[q_1, q_2, \dots]^\wedge = \bigoplus_{n=0}^{\infty} V_n$$

$$V_n = \{f \in \mathbb{C}[q_1, \dots, q_n] \mid \deg f = n\}, \quad \text{assuming } \deg q_i = i.$$

where the generator q_n acts by multiplication by q_n and p_n by derivation

$$p_n = n \frac{\partial}{\partial q_n}.$$

The algebra of free fermions has two infinite sequences of generators ψ_k and ψ_k^* labeled by half-integers. The defining relations are given in terms of anti-commutators

$$\{a, b\} = ab + ba.$$

They read

$$\{\psi_i, \psi_j^*\} = \delta_{ij}, \quad \{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0. \quad (2.2)$$

Fermionic Fock space \mathcal{F} is spanned by vectors obtained by action of finite products of the generators on the vacuum vector $|0\rangle$. It is assumed that

$$\psi_{-k}|0\rangle = 0, \quad \psi_k^*|0\rangle = 0 \quad \text{for } k > 0.$$

The generators ψ_{-k} and ψ_k^* for $k > 0$ are called annihilation operators; the generators ψ_k and ψ_{-k}^* are creation operators. For any product of the fermionic generators the normal order is defined by moving all annihilation operators on the right. For example, the normal ordering of the product $\psi_i \psi_i^*$ is

$$:\psi_i \psi_i^* := \begin{cases} \psi_i \psi_i^*, & i > 0 \\ -\psi_i^* \psi_i, & i < 0 \end{cases}.$$

Vectors of charge zero in the fermionic Fock space can be written as linear combinations of vectors of the form

$$\psi_{i_1} \dots \psi_{i_n} \psi_{j_1}^* \dots \psi_{j_n}^* |0\rangle$$

for an arbitrary integer $n \geq 0$. The subspace in the Fock space spanned by these vectors will be denoted \mathcal{F}_0 . The dual space \mathcal{F}^* is obtained by a similar action on the bra-vector $\langle 0|$ assuming that

$$0 = \langle 0| \psi_k, \quad 0 = \langle 0| \psi_{-k}^* \quad \text{for } k > 0.$$

The pairing $\mathcal{F}^* \times \mathcal{F} \rightarrow \mathbb{C}$ is completely defined by the anticommutation relations (2.2) along with the normalization

$$\langle 0|0\rangle = 1.$$

Boson-fermion correspondence is an isomorphism of linear spaces

$$\Phi : \mathcal{F}_0 \rightarrow \mathbb{C}[q_1, q_2, \dots]^\wedge$$

given by the formula

$$\Phi(\xi |0\rangle) = \langle 0| e^{K(q)} \xi |0\rangle \quad (2.3)$$

$$K(q) = \sum_{n=1}^{\infty} \frac{q_n}{n} \sum_j : \psi_j \psi_{j+n}^* :$$

for any state $\xi |0\rangle \in \mathcal{F}_0$.

The following formulae hold true

$$\begin{aligned} e^{K(q)} \psi_i e^{-K(q)} &= \psi_i + h_1(q) \psi_{i-1} + h_2(q) \psi_{i-2} + \dots \\ e^{K(q)} \psi_i^* e^{-K(q)} &= \psi_i^* + h_1(-q) \psi_{i+1}^* + h_2(-q) \psi_{i+2}^* + \dots \end{aligned} \quad (2.4)$$

Here $h_k(q)$ are elementary Schur polynomials defined in (1.9).

Due to the isomorphism (2.3) any operator on \mathcal{F}_0 becomes an operator on the bosonic Fock space $\mathbb{C}[q_1, q_2, \dots]^\wedge$ and vice versa. For example, the pull-back of the operators q_n and p_n acting on the bosonic Fock space is given by the following infinite sums

$$\begin{aligned} \Phi^{-1} q_n \Phi &= \sum_{j \in \mathbb{Z} + \frac{1}{2}} : \psi_j \psi_{j-n}^* : \\ \Phi^{-1} p_n \Phi &= \sum_{j \in \mathbb{Z} + \frac{1}{2}} : \psi_j \psi_{j+n}^* : \end{aligned}$$

The following statement was proved by P. Rossi in [20].

Proposition 2.1. *The fermionic representation of the Hamiltonian (1.8) reads*

$$\Phi^{-1} \hat{\mathcal{H}}(z, u_0, \hbar = 1) \Phi = e^{z u_0} \left(z \sum_{k \in \mathbb{Z} + \frac{1}{2}} e^{kz} : \psi_k \psi_k^* : + \frac{1}{s(z)} \right). \quad (2.5)$$

Following [2] construct a basis of the fermionic Fock space \mathcal{F}_0 labeled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0)$ of arbitrary length $n \geq 0$. It is convenient to use Frobenius notations for partitions

$$\lambda = (\alpha_1, \dots, \alpha_d \mid \beta_1, \dots, \beta_d) \quad (2.6)$$

where $d = d(\lambda)$ is the length of the diagonal of the Young tableau λ , the numbers α_i, β_i are defined as follows

$$\alpha_i = \lambda_i - i, \quad \beta_i = \lambda'_i - i, \quad i = 1, \dots, d(\lambda).$$

Here λ' is the transposed Young tableau. Define the state $|\lambda\rangle \in \mathcal{F}_0$ by

$$|\lambda\rangle = \psi_{\alpha_1 + \frac{1}{2}} \psi_{\alpha_2 + \frac{1}{2}} \dots \psi_{\alpha_{d(\lambda)} + \frac{1}{2}} \psi_{-\beta_{d(\lambda)} - \frac{1}{2}}^* \dots \psi_{-\beta_2 - \frac{1}{2}}^* \psi_{-\beta_1 - \frac{1}{2}}^* |0\rangle. \quad (2.7)$$

For empty partition put $|\emptyset\rangle = |0\rangle$.

The following statement readily follows from Proposition 2.10 of [2].

Proposition 2.2. *For any partition λ one has*

$$\Phi(|\lambda\rangle) = (-1)^{b(\lambda)} s_\lambda(q) \quad (2.8)$$

where

$$b(\lambda) = \sum_{i=1}^{d(\lambda)} (\beta_i + 1).$$

Denote

$$\mathcal{O}(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} e^{kz} : \psi_k \psi_k^* : \quad (2.9)$$

the nontrivial part of the operator (2.5).

Lemma 2.3. *The states $|\lambda\rangle$ are eigenvectors of the Hamiltonian (2.9)*

$$\mathcal{O}(z)|\lambda\rangle = \sum_{i=1}^{d(\lambda)} \left[e^{z(\alpha_i + \frac{1}{2})} - e^{-z(\beta_i + \frac{1}{2})} \right] |\lambda\rangle. \quad (2.10)$$

Here the partition λ is represented in the Frobenius form (2.6).

Proof. One has

$$\begin{aligned} \mathcal{O}(z)|\lambda\rangle &= \sum_{k>0} e^{kz} \psi_k \psi_k^* \psi_{\alpha_1 + \frac{1}{2}} \psi_{\alpha_2 + \frac{1}{2}} \cdots \psi_{\alpha_{d(\lambda)} + \frac{1}{2}} \psi_{-\beta_{d(\lambda)} - \frac{1}{2}}^* \cdots \psi_{-\beta_2 - \frac{1}{2}}^* \psi_{-\beta_1 - \frac{1}{2}}^* |0\rangle \\ &- \sum_{k>0} e^{-kz} \psi_{-k}^* \psi_{-k} \psi_{\alpha_1 + \frac{1}{2}} \psi_{\alpha_2 + \frac{1}{2}} \cdots \psi_{\alpha_{d(\lambda)} + \frac{1}{2}} \psi_{-\beta_{d(\lambda)} - \frac{1}{2}}^* \cdots \psi_{-\beta_2 - \frac{1}{2}}^* \psi_{-\beta_1 - \frac{1}{2}}^* |0\rangle. \end{aligned}$$

Since ψ_k^* annihilates the vacuum for $k > 0$, in the first sum the only nonzero terms are those for which $k = \alpha_i + \frac{1}{2}$ for some $1 \leq i \leq d(\lambda)$. So

$$\begin{aligned} &\sum_{k>0} e^{kz} \psi_k \psi_k^* \psi_{\alpha_1 + \frac{1}{2}} \psi_{\alpha_2 + \frac{1}{2}} \cdots \psi_{\alpha_{d(\lambda)} + \frac{1}{2}} \psi_{-\beta_{d(\lambda)} - \frac{1}{2}}^* \cdots \psi_{-\beta_2 - \frac{1}{2}}^* \psi_{-\beta_1 - \frac{1}{2}}^* |0\rangle \\ &= \sum_{i=1}^{d(\lambda)} e^{z(\alpha_i + \frac{1}{2})} |\lambda\rangle. \end{aligned}$$

In a similar way in the second sum the nonzero terms come from $k = \beta_i + \frac{1}{2}$, $1 \leq i \leq d(\lambda)$. This gives the second half of eq. (2.10). \square

It is easy to see that

$$\sum_{i=1}^{d(\lambda)} \left[e^{z(\alpha_i + \frac{1}{2})} - e^{-z(\beta_i + \frac{1}{2})} \right] = \sum_{i=1}^{l(\lambda)} \left[e^{z(\lambda_i - i + \frac{1}{2})} - e^{z(-i + \frac{1}{2})} \right]. \quad (2.11)$$

So, to complete the proof of the theorem it remains to reinsert the Planck constant. This can be done by rescaling

$$q_k \mapsto \hbar^{\frac{k-1}{2}} q_k, \quad z \mapsto \hbar^{\frac{1}{2}} z.$$

The theorem is proved.

Corollary 2.4. *For any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l(\lambda)} > 0) = (\alpha_1, \dots, \alpha_{d(\lambda)} | \beta_1, \dots, \beta_{d(\lambda)})$ the polynomials $s_\lambda \left(\frac{q}{\hbar^{1/2}} \right)$ are common eigenvectors of the commuting operators \hat{H}_k*

$$\hat{H}_k s_\lambda \left(\frac{q}{\hbar^{1/2}} \right) = E_k(\lambda, u_0, \hbar) s_\lambda \left(\frac{q}{\hbar^{1/2}} \right), \quad k \geq -1 \quad (2.12)$$

with the eigenvalues given by one of the two equivalent expressions

$$\begin{aligned}
 E_k(\lambda, u_0, \hbar) &= c_k(u_0, \hbar) \\
 &+ \hbar^{\frac{1}{2}} \sum_{i=1}^{l(\lambda)} \frac{\left[u_0 + \hbar^{\frac{1}{2}} \left(\lambda_i - i + \frac{1}{2} \right) \right]^{k+1} - \left[u_0 + \hbar^{\frac{1}{2}} \left(-i + \frac{1}{2} \right) \right]^{k+1}}{(k+1)!} \\
 &= c_k(u_0, \hbar) + \hbar^{\frac{1}{2}} \sum_{i=1}^{d(\lambda)} \frac{\left[u_0 + \hbar^{\frac{1}{2}} \left(\alpha_i + \frac{1}{2} \right) \right]^{k+1} - \left[u_0 - \hbar^{\frac{1}{2}} \left(\beta_i + \frac{1}{2} \right) \right]^{k+1}}{(k+1)!}
 \end{aligned} \tag{2.13}$$

where

$$c_k(u_0, \hbar) = -\frac{1}{(k+2)!} \sum_{j=0}^{k+2} \binom{k+2}{j} (1 - 2^{1-j}) B_j \hbar^{\frac{j}{2}} u_0^{k-j+2},$$

B_j are Bernoulli numbers.

Proof. With the help of an obvious formula

$$\sum_{i \geq 1} e^{z(-i + \frac{1}{2})} = \frac{1}{z s(z)}, \quad \operatorname{Re} z > 0$$

the expressions (1.10) can be rewritten in the form

$$\begin{aligned}
 E(z, \lambda, u_0, \hbar) &= e^{z u_0} \left[\frac{1}{s\left(\hbar^{\frac{1}{2}} z\right)} + \hbar^{\frac{1}{2}} z \sum_{i=1}^{l(\lambda)} \left(e^{z \hbar^{\frac{1}{2}} \left(\lambda_i - i + \frac{1}{2} \right)} - e^{z \hbar^{\frac{1}{2}} \left(-i + \frac{1}{2} \right)} \right) \right] \\
 &= e^{z u_0} \left[\frac{1}{s\left(\hbar^{\frac{1}{2}} z\right)} + \hbar^{\frac{1}{2}} z \sum_{i=1}^{d(\lambda)} \left(e^{z \hbar^{\frac{1}{2}} \left(\alpha_i + \frac{1}{2} \right)} - e^{-z \hbar^{\frac{1}{2}} \left(\beta_i + \frac{1}{2} \right)} \right) \right].
 \end{aligned}$$

Expanding the right-hand side in z and using the Taylor expansion

$$\frac{1}{s(t)} = \sum_{n=0}^{\infty} (2^{1-n} - 1) \frac{B_n}{n!} t^n$$

one arrives at the needed formula for the eigenvalues. \square

Remark 2.5. An alternative derivation [18] of the spectrum of the operator (2.9) uses a realization of the fermionic algebra by operators acting on the Sato infinite-dimensional Grassmannian. Let us briefly outline this construction that will be useful below. Introduce an infinite-dimensional space with a basis e_k ,

$$\mathcal{V} = \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} \mathbb{C} e_k.$$

Denote

$$\begin{aligned}
 \Lambda^{\frac{\infty}{2}} \mathcal{V} &= \operatorname{span} (e_{i_1} \wedge e_{i_2} \wedge \dots \mid i_1 > i_2 > \dots, \quad i_k = -k \text{ for large } k), \\
 e_j \wedge e_i &= -e_i \wedge e_j.
 \end{aligned} \tag{2.14}$$

Action of the fermionic algebra is defined by adding or erasing a factor on the left of a semi-infinite wedge product

$$\psi_k = e_k \wedge, \quad \psi_k^* = \frac{\partial}{\partial e_k}, \quad k \in \mathbb{Z} + \frac{1}{2}. \quad (2.15)$$

The vacuum vector reads

$$|0\rangle = e_{-1/2} \wedge e_{-3/2} \wedge \dots$$

For any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0)$ consider

$$v_\lambda = e_{\lambda_1 - \frac{1}{2}} \wedge e_{\lambda_2 - \frac{3}{2}} \wedge \dots \wedge e_{\lambda_l - l + \frac{1}{2}} \wedge e_{-l - \frac{1}{2}} \wedge \dots \in \Lambda^{\frac{\infty}{2}} \mathcal{V}. \quad (2.16)$$

The vectors of the form (2.16) span the fermionic phase space \mathcal{F}_0 . According to [18] it is an eigenvector of the operator (2.9) with the eigenvalue

$$\sum_{i=1}^l \left[e^{z(\lambda_i - i + \frac{1}{2})} - e^{z(-i + \frac{1}{2})} \right].$$

3. SFT potential of a disk

Let us very briefly outline the main constructions of SFT; more details can be found in [9], [6], [8]; see also [20].

Let (V, α) be a contact manifold with a contact form α . Denote R the Reeb vector field on V . Let $\mathcal{P}(V)$ be the set of periodic orbits of R . For simplicity assume that all periodic orbits are non-degenerate; in this case $\mathcal{P}(V)$ is a discrete set. The first playing character is the *SFT Hamiltonian* \mathbf{H}^V . The construction uses intersection theory on the moduli spaces of pseudo-holomorphic maps

$$f : (C_g, x_1, \dots, x_n; y_1^-, \dots, y_{r_-}^-, y_1^+, \dots, y_{r_+}^+) \rightarrow V \times \mathbb{R}$$

of Riemann surfaces of genus g with n marked points x_1, \dots, x_n , r_- negative punctures $y_1^-, \dots, y_{r_-}^- \in C_g$ and r_+ positive punctures $y_1^+, \dots, y_{r_+}^+ \in C_g$. One has to choose an almost complex structure J on the cylinder $V \times \mathbb{R}$ compatible with the symplectic form $d(e^t \alpha)$ where $t \in \mathbb{R}$. It must be invariant with respect to t -translations and satisfy certain additional restrictions. The boundary conditions for the pseudoholomorphic maps near the positive/negative punctures depend on a choice of suitably oriented Reeb orbits $\gamma_1^+, \dots, \gamma_{r_+}^+$ and $\gamma_1^-, \dots, \gamma_{r_-}^-$ in $\mathcal{P}(V)$. Namely, the map f must be asymptotically cylindrical near the negative punctures $y_1^-, \dots, y_{r_-}^-$ to the orbits $\gamma_1^-, \dots, \gamma_{r_-}^-$ at $t \rightarrow -\infty$ and to $\gamma_1^+, \dots, \gamma_{r_+}^+$ near the positive punctures $y_1^+, \dots, y_{r_+}^+$ at $t \rightarrow +\infty$. Denote

$$M_{g,n}(V; \gamma^-, \gamma^+), \quad \gamma^+ = (\gamma_1^-, \dots, \gamma_{r_-}^-), \quad \gamma^- = (\gamma_1^+, \dots, \gamma_{r_+}^+) \quad (3.1)$$

the moduli space of such maps. The group \mathbb{R} acts on the moduli space by translations. The quotient can be compactified by adding stable holomorphic buildings [6]. Denote $\overline{M}_{g,n}(V; \gamma^-, \gamma^+)/\mathbb{R}$ the compactified space. Choosing a system of closed differential forms $\theta_1, \dots, \theta_N$ on V define the so-called *SFT*

Hamiltonian $\mathbf{H}^V = \mathbf{H}^V(\mathbf{t}, \mathbf{q}, \mathbf{p}, \hbar)$ as a generating function of the intersection numbers on the moduli spaces as follows⁸

$$\begin{aligned} \mathbf{H}^V = & \sum_{g \geq 0} \hbar^g \sum_{n, r_-, r_+} \sum_{\substack{0 \leq i_1, \dots, i_n \leq N \\ 0 \leq k_1, \dots, k_n}} \sum_{\substack{\gamma_{b_1}^-, \dots, \gamma_{b_{r_-}}^- \\ \gamma_{c_1}^+, \dots, \gamma_{c_{r_+}}^+}} \frac{t_{k_1}^{i_1} \dots t_{k_n}^{i_n}}{n!} \frac{q_{b_1} \dots q_{b_{r_-}}}{r_-!} \frac{p_{c_1} \dots p_{c_{r_+}}}{r_+!} \\ & \times \frac{\int \text{ev}_1^*(\theta_{i_1}) \psi_1^{k_1} \wedge \dots \wedge \text{ev}_n^*(\theta_{i_n}) \psi_n^{k_n}}{\overline{M_{g,n}(V; \gamma^-, \gamma^+) / \mathbb{R}}} \end{aligned} \quad (3.2)$$

Here ev_i are the evaluation maps

$$\text{ev}_i : \overline{M_{g,n}(V; \gamma^-, \gamma^+) / \mathbb{R}} \rightarrow V, \quad f \mapsto f(x_i), \quad i = 1, \dots, n,$$

$\psi_i = c_1(\mathcal{L}_i)$ are the Chern classes of the tautological line bundles over the moduli spaces. All the independent variables in the formula are $\mathbb{Z}/2$ -graded. In particular, the parity of the variables t_k^i , $i = 1, \dots, N$, $k = 0, 1, \dots$ is determined by parity of the chosen differential forms $\theta_1, \dots, \theta_N$. The q - and p -variables labeled by Reeb orbits belong to a Weyl (super)-algebra with the commutation relations

$$[p_\gamma, q_{\gamma'}]_{\pm} = \hbar \text{mult}(\gamma) \delta_{\gamma \gamma'}.$$

As the moduli spaces $\overline{M_{g,n}(V; \gamma^-, \gamma^+) / \mathbb{R}}$ are odd-dimensional, the generating function is an odd element of the algebra. Expanding \mathbf{H}^V with respect to *odd* variables t_k^i and taking the cohomology class one obtains therefore infinite families of commuting Hamiltonians

$$\hat{H}_{i,k}^V(\mathbf{q}, \mathbf{p}, \hbar) := \frac{\partial}{\partial t_k^i} [\mathbf{H}^V]_{\mathbf{t}=0}$$

in the Weyl algebra.

In the simplest case $V = S^1$ the above construction yields the commuting quantum Hamiltonians (1.8). In this case the set of Reeb orbits is just the set of integers. The moduli spaces (3.1) are nontrivial only for positive multiplicities. Thus one has independent variables q_n, p_n , $n = 1, 2, \dots$ satisfying the commutation relation (1.5). Choose $\theta_0 = 1$, $\theta_1 = d\varphi$ as two differential forms on S^1 (here φ is the angular coordinate on the circle). The SFT Hamiltonian \mathbf{H}^{S^1} is an odd element of the Weyl algebra with generators p_n, q_n , $n \geq 1$ depending on even variables t_k^0 and odd variables t_k^1 , $k \geq 0$. Taking derivatives with respect to odd variables produces the commuting quantum Hamiltonians according to the following procedure [8].

Proposition 3.1. *Restriction of the derivatives of $\mathbf{H}^{S^1} = \mathbf{H}^{S^1}(\mathbf{t}, q, p, \hbar)$ onto the locus $t_0^0 = u_0$, $t_k^0 = 0$ for $k > 0$, $t_k^1 = 0$ for all k yields the commuting*

⁸Our definition differs by a factor \hbar from the one of [9], [8]. This factor will be restored below, see the Schrödinger equation (3.9).

Hamiltonians (1.8)

$$\frac{\partial}{\partial t_n^1} \mathbf{H}^{S^1} \big|_{t_0^0=u_0, \quad t_k^0=0 \text{ for } k>0, \quad t_k^1=0} = \hat{H}_n(u_0, \hbar), \quad n \geq 0. \quad (3.3)$$

More generally, with a symplectic cobordism W between two contact manifolds V_+ and V_- ,

$$\partial W = V_+ \cup (-V_-)$$

one can associate a family of moduli spaces $M_{g,n}(W, \gamma^-, \gamma^+)$ labeled by collections of Reeb orbits $\gamma^- = (\gamma_1^-, \dots, \gamma_{r_-}^-)$ on V_- and $\gamma^+ = (\gamma_1^+, \dots, \gamma_{r_+}^+)$ on V_+ . To this end one has to consider pseudoholomorphic maps with cylindrical ends of Riemann surfaces of genus g with r_- negative and r_+ positive punctures and also with n marked points x_1, \dots, x_n to the symplectic manifold $W \cup (V_- \times (-\infty, 0]) \cup (V_+ \times [0, +\infty))$ equipped with an appropriate almost complex structure. Evaluation maps are defined

$$\text{ev}_i : M_{g,n}(W, \gamma^-, \gamma^+) \rightarrow W, \quad i = 1, \dots, n$$

by taking images of the marked points. In this way one arrives at the SFT potential

$$Z_{\text{SFT}}^W(\mathbf{t}, \mathbf{q}, \mathbf{p}, \hbar) = e^{\frac{1}{\hbar} \sum_{g \geq 0} \hbar^g \mathcal{F}_g^{\text{SFT}}(\mathbf{t}, \mathbf{q}, \mathbf{p})}. \quad (3.4)$$

Here the genus g part $\mathcal{F}_g^{\text{SFT}}(\mathbf{t}, \mathbf{q}, \mathbf{p})$ is the generating function of numbers obtained by integrating over the compactified moduli spaces $\overline{M}_{g,n}(W, \gamma^-, \gamma^+)$ the pull-backs $\text{ev}_i^*(\phi)$ of differential forms ϕ on W along with ψ -classes. The independent variables $\mathbf{q} = (q_{\gamma^-})$ and $\mathbf{p} = (p_{\gamma^+})$ are labeled by Reeb orbits on V_- and V_+ respectively.

If W is obtained by glueing together two symplectic cobordisms

$$W = W_1 \cup W_2, \quad \partial W_1 = V \cup (-V_-), \quad \partial W_2 = V_+ \cup (-V)$$

between contact manifold V_- and V and V and V_+ respectively then the SFT potential of W can be obtained by applying the following glueing rules

$$\begin{aligned} Z^W(\mathbf{q}_-, \mathbf{p}_+) &= Z^{W_1}(\mathbf{q}_-, \vec{\mathbf{p}}) Z^{W_2}(\mathbf{q}, \mathbf{p}_+) \big|_{\mathbf{q}=0} \\ &= Z^{W_1}(\mathbf{q}_-, \mathbf{p}) Z^{W_2}(\overleftarrow{\mathbf{q}}, \mathbf{p}_+) \big|_{\mathbf{p}=0}. \end{aligned} \quad (3.5)$$

The variables $\mathbf{q} = (q_\gamma)$ and $\mathbf{p} = (p_\gamma)$ are labeled by Reeb orbits on V . The left-acting operator \vec{p}_γ and right-acting operator \overleftarrow{q}_γ are defined by

$$\vec{p}_\gamma = \hbar \text{mult}(\gamma) \frac{\partial}{\partial q_\gamma}, \quad \overleftarrow{q}_\gamma = \hbar \text{mult}(\gamma) \frac{\partial}{\partial p_\gamma}.$$

In the particular case of a symplectic manifold W with a boundary $V = V_-$ the total SFT potential depends only on the canonical coordinates q_γ labeled by Reeb orbits on V . It can be considered as a state in the bosonic Fock space generated by Reeb orbits on V . This state will be denoted by

$$\Psi_{\text{SFT}}^W(\mathbf{t}, \mathbf{q}, \hbar) := Z_{\text{SFT}}^W(\mathbf{t}, \mathbf{q}, \hbar).$$

In a similar way, for a symplectic manifold W with a boundary $V = V_+$ one obtains a state in the dual Fock space

$$\Psi_{\text{SFT}}^W(\mathbf{t}, \mathbf{p}, \hbar) := Z_{\text{SFT}}^W(\mathbf{t}, \mathbf{p}, \hbar).$$

Finally, for a closed symplectic manifold W the total SFT potential coincides with the GW total descendent potential

$$Z_{\text{SFT}}^W(\mathbf{t}, \hbar) = Z_{\text{GW}}^W(\mathbf{t}, \hbar).$$

To insert pullbacks of cohomology classes of the form $\phi = \theta \wedge d\rho$, $\theta \in \Omega^{\text{odd}}(V_+)$, $d\rho \in \Omega_{\text{comp}}^1(\mathbb{R})$ along with their descendents one can use the SFT Hamiltonians (3.2):

$$Z_{\text{SFT}}^W(\mathbf{t}, \mathbf{q}_-, \mathbf{p}_+, \hbar) = \left(Z_{\text{SFT}}^W(\mathbf{q}_-, \mathbf{p}, \hbar) e^{\frac{1}{\hbar} \sum_{k \geq 0} t_k^\phi \hat{H}_k^{V_+}(\mathbf{q}_-, \mathbf{p}_+, \hbar)} \right)_{\mathbf{p}=0} \quad (3.6)$$

where the quantum Hamiltonians $H_k^{V_+}$ are defined in a way similar to (3.3),

$$\hat{H}_k^{V_+}(\mathbf{q}, \mathbf{p}, \hbar) = \frac{\partial}{\partial t_k^\theta} \mathbf{H}^{V_+}(\mathbf{t}, \mathbf{q}, \mathbf{p}, \hbar)|_{\mathbf{t}=0}, \quad k \geq 0.$$

In the calculations of this section the following two formulae from the theory of Schur polynomials will be useful (see, e.g., the Macdonald book [14]).

Proposition 3.2. *The following formulae hold true*

$$q_1^n = \sum_{|\lambda|=n} \dim \lambda \cdot s_\lambda(q_1, \dots, q_n) \quad (3.7)$$

$$s_\lambda(q_1, \dots, q_n) = \frac{\dim \lambda}{n!} q_1^n + \dots, \quad n = |\lambda| \quad (3.8)$$

where periods stand for monomials of lower degree in q_1 .

In these formulae $\dim \lambda$ is the dimension of the irreducible representation of the symmetric group S_n associated with the Young tableau λ . Recall that this dimension can be computed using the hook length formula

$$\dim \lambda = \frac{|\lambda|!}{\prod_{(i,j) \in \lambda} h(i,j)}$$

(*ibid.*)

We are now ready to prove Theorem 1.5.

Proof. According to the prescription (3.6) one has to solve a system of pairwise commuting non-stationary Schrödinger equations for the wave function $\Psi = \Psi_{\text{SFT}}^{\text{disk}}$

$$\hbar \frac{\partial}{\partial t_k} \Psi = \Psi \overleftarrow{H}_k, \quad k = 0, 1, 2, \dots \quad (3.9)$$

with the plane wave initial data

$$\Psi|_{\mathbf{t}=0} = e^{\frac{p_1}{\hbar}}.$$

Using eq. (3.7) one decomposes the initial condition in a linear combination of common eigenvectors of the commuting Hamiltonians

$$e^{\frac{p_1}{\hbar}} = \sum_{\lambda} \hbar^{-\frac{|\lambda|}{2}} \frac{\dim \lambda}{|\lambda|!} s_\lambda \left(\frac{p}{\hbar^{1/2}} \right).$$

With the help of such a decomposition, using (2.13), one arrives at (1.13). \square

We will now prove Corollary 1.7 .

Proof. According to [16] tau-functions of the KP hierarchy correspond to simple multivectors

$$\tau(q_1, q_2, \dots) = \Phi \left(\varphi_{-\frac{1}{2}} \wedge \varphi_{-\frac{3}{2}} \wedge \dots \right), \quad \varphi_\nu \in \mathcal{V}, \quad \nu \in \mathbb{Z} + \frac{1}{2}, \quad \nu < 0$$

(for notations see Remark 2.5 above). Choosing

$$\varphi_\nu = \sum_{i \geq 0} A_{\nu, i} e_{\nu+i}$$

$$A_{\nu, i} = \frac{1}{i!} \exp \left\{ \hbar^{\frac{1}{2}} \sum_{k \geq 0} t_k \frac{[u_0 + \hbar^{\frac{1}{2}}(\nu + i)]^{k+1} - [u_0 + \hbar^{\frac{1}{2}}\nu]^{k+1}}{(k+1)!} \right\}$$

(cf. [12]) one obtains the expression (1.13). \square

Proof of Corollary 1.8. According to the above procedures to compute the Gromov–Witten potential of \mathbf{P}^1 within the stationary sector one has to apply the SFT potential of the disk to the potential of the cap without insertions, i.e.,

$$Z_{\mathbf{P}^1}^\bullet(\mathbf{t}, z, \hbar) = \Psi_{\text{SFT}}^{\text{disk}}(u_0, \mathbf{t}, \vec{\mathbf{p}}; \hbar) e^{\frac{z q_1}{\hbar}}|_{q=0}.$$

Here the independent parameter z is added for convenience in order to separate terms of various degrees. Due to (3.8) one has

$$s_\lambda(\vec{\mathbf{p}}) q_1^n|_{q=0} = \begin{cases} \hbar^n \dim \lambda, & |\lambda| = n \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof.

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